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The structure of the spectrum of a system of difference equations

Elgiz Bairamov, Cafer Coskun*

Department of Mathematics, Ankara University, 06100 Tandoğan, Ankara, Turkey

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Abstract

In this paper, we investigate the structure of the discrete spectrum of the system of non-selfadjoint difference equations of first order using the uniqueness theorems of analytic functions. We also obtained the sufficient conditions on coefficients of this system under which its discrete spectrum is finite.

Keywords: Difference equations; Spectral analysis; Discrete spectrum; Spectral singularities

1. Introduction

Various problems of spectral analysis of selfadjoint difference equations have already been investigated in detail in [1,2], where one can find a large list of references on the subject.

In recent years some problems of spectral analysis of non-selfadjoint difference equations with continuous and discrete spectrum have been studied by some authors [3,4]. It is known that one of the most important properties of non-selfadjoint differential equations is that of having spectral singularities [5–11]. In [12] it is proved by examples that non-selfadjoint difference equations of second order have spectral singularities. So the theory of these equations becomes interesting. Some problems of spectral analysis of difference equations with spectral singularities have been studied in [13,14].

Let us consider the system of difference equations of first order

* Corresponding author.

E-mail addresses: bairamov@science.ankara.edu.tr (E. Bairamov), coskun@science.ankara.edu.tr (C. Coskun).

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$$\begin{cases} a_{n+1} y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ b_n y_n^{(1)} + a_{n-1} y_{n-1}^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)}, \qquad n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}, \end{cases} \tag{1.1}$$

where $\left\{ \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} \right\}_{n\in\mathbb{Z}}$ are vector sequences, $\{a_n\}_{n\in\mathbb{Z}}$, $\{b_n\}_{n\in\mathbb{Z}}$, $\{p_n\}_{n\in\mathbb{Z}}$ and $\{q_n\}_{n\in\mathbb{Z}}$ are complex sequences, $a_n \neq 0$, $b_n \neq 0$ for all $n \in \mathbb{Z}$ and λ is a spectral parameter.

If for all $n \in \mathbb{Z}$, $a_n \equiv 1$ and $b_n \equiv -1$ then the system (1.1) reduces to

$$\begin{cases} \Delta y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ -\Delta y_n^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)}, \qquad n \in \mathbb{Z} \end{cases} \tag{1.2}$$

where Δ is a forward difference operator, i.e., $\Delta u_n = u_{n+1} - u_n$. The system (1.2) is the discrete analogue of the well-known Dirac system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

(see [15], ch. 2). Therefore the system (1.2) is called the discrete Dirac system.

In this paper which is a continuation of [16], we aim to investigate the structure of the eigenvalues and spectral singularities of (1.1) (also (1.2)) using the uniqueness theorems of analytic functions. Also the sufficient conditions on coefficients of the systems (1.1) and (1.2) under which their eigenvalues and spectral singularities are finite have been obtained which is the solution of the open problem given in [16].

2. Jost solutions of (1.1)

Let $\{a_n\}_{n\in\mathbb{Z}}$, $\{b_n\}_{n\in\mathbb{Z}}$, $\{p_n\}_{n\in\mathbb{Z}}$ and $\{q_n\}_{n\in\mathbb{Z}}$ satisfy

$$\sum_{n\in\mathbb{Z}} |n|(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|) < \infty. \tag{2.1}$$

It is well known that [16], under condition (2.1) Eq. (1.1) has the solutions

$$\begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} = \alpha_n \left(E_2 + \sum_{m=1}^{\infty} A_{nm} e^{imz} \right) \begin{pmatrix} e^{iz/2} \\ -i \end{pmatrix} e^{inz}, \qquad n \in \mathbb{Z},$$

$$\begin{pmatrix} g_n^{(1)}(z) \\ g_n^{(2)}(z) \end{pmatrix} = \beta_n \left(E_2 + \sum_{-\infty}^{m=-1} B_{nm} e^{-imz} \right) \begin{pmatrix} -i \\ e^{iz/2} \end{pmatrix} e^{-inz}, \qquad n \in \mathbb{Z},$$

for $\lambda = 2 \sin \frac{z}{2}$ and $\overline{\mathbb{C}}_+ := \{z : z \in \mathbb{C}, \operatorname{Im} z \geq 0\}$, where

$$\alpha_n = \begin{pmatrix} \alpha_n^{11} & \alpha_n^{12} \\ \alpha_n^{21} & \alpha_n^{22} \end{pmatrix}, \qquad \beta_n = \begin{pmatrix} \beta_n^{11} & \beta_n^{12} \\ \beta_n^{21} & \beta_n^{22} \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A_{nm} = \begin{pmatrix} A_{nm}^{11} & A_{nm}^{12} \\ A_{nm}^{21} & A_{nm}^{22} \end{pmatrix}, \qquad B_{nm} = \begin{pmatrix} B_{nm}^{11} & B_{nm}^{12} \\ B_{nm}^{21} & B_{nm}^{22} \end{pmatrix}.$$

Note that α_n^{ij}, β_n^{ij}, A_{nm}^{ij} and B_{nm}^{ij} $(i, j = 1, 2)$ are expressed in terms of $\{a_n\}_{n\in\mathbb{Z}}$, $\{b_n\}_{n\in\mathbb{Z}}$, $\{p_n\}_{n\in\mathbb{Z}}$ and $\{q_n\}_{n\in\mathbb{Z}}$. Moreover

$$|A_{nm}^{ij}| \le C \sum_{k=n+[m/2]}^{\infty} (|1-a_k| + |1+b_k| + |p_k| + |q_k|), \tag{2.2}$$

$$|B_{nm}^{ij}| \le C \sum_{-\infty}^{k=n+[m/2]+1} (|1-a_k| + |1+b_k| + |p_k| + |q_k|), \qquad i, j = 1, 2. \tag{2.3}$$

hold, where $[m/2]$ is the integer part of $m/2$ and $C > 0$ is a constant [16]. Therefore f_n^j and $g_n^j (i = 1, 2, n \in \mathbb{Z})$ are analytic with respect to z in $\mathbb{C}_+ := \{z : z \in \mathbb{C}, \operatorname{Im} z > 0\}$ and continuous up to the real axis. The solutions

$$f(z) = \left\{ \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} \right\}_{n\in\mathbb{Z}} \qquad \text{and} \qquad g(z) = \left\{ \begin{pmatrix} g_n^{(1)}(z) \\ g_n^{(2)}(z) \end{pmatrix} \right\}_{n\in\mathbb{Z}}$$

are called Jost solutions of (1.1).

The Wronskian of the solutions

$$y(\lambda) := \left\{ \begin{pmatrix} y_n^{(1)}(\lambda) \\ y_n^{(2)}(\lambda) \end{pmatrix} \right\}_{n\in\mathbb{Z}} \qquad \text{and} \qquad u(\lambda) := \left\{ \begin{pmatrix} u_n^{(1)}(\lambda) \\ u_n^{(2)}(\lambda) \end{pmatrix} \right\}_{n\in\mathbb{Z}}$$

of (1.1) is defined by

$$W[y(\lambda), u(\lambda)] = a_n \left[y_n^{(1)}(\lambda) u_{n+1}^{(2)}(\lambda) - y_{n+1}^{(2)}(\lambda) u_n^{(1)}(\lambda) \right].$$

3. Discrete spectrum of (1.1)

If we define

$$w(z) := W[f(z), g(z)],$$

then w is analytic in $\mathbb{C}_+$, continuous up to the real axis and

$$w(z) = w(z + 4\pi).$$

Let

$$P_0 := \{z : z \in \mathbb{C},\ 0 \le \operatorname{Re} z \le 4\pi,\ \operatorname{Im} z > 0\},$$
$$P := \{z : z \in \mathbb{C},\ 0 \le \operatorname{Re} z \le 4\pi,\ \operatorname{Im} z \ge 0\}.$$

We will denote the set of all eigenvalues and spectral singularities of Eq. (1.1) by σ_d and σ_{ss}, respectively. It is obvious that

$$\sigma_d = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2},\ z \in P_0,\ w(z) = 0 \right\}, \tag{3.1}$$

$$\sigma_{ss} = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2},\ z \in [0, 4\pi],\ w(z) = 0 \right\}, \tag{3.2}$$

and

$$w(z) = a_0 [f_0^{(1)}(z) g_1^{(2)}(z) - f_1^{(2)}(z) g_0^{(1)}(z)]$$
$$= \left[\prod_{n\in\mathbb{Z}} (-1)^n a_n b_n \right]^{-1} \left\{ \left[1 + \sum_{m=1}^{\infty} \left(A_{0m}^{11} - \mathrm{i} A_{0m}^{12} \mathrm{e}^{-\mathrm{i}z/2} \right) \mathrm{e}^{\mathrm{i}mz} \right] \right.$$

$$\times \left[1 + \sum_{-\infty}^{m=-1} \left(B_{1m}^{22} - \mathrm{i} B_{1m}^{21} \mathrm{e}^{-\mathrm{i}z/2} \right) \mathrm{e}^{-\mathrm{i}mz} \right] - a_0 \left[\left(p_1 - A_{11}^{12} \right) \mathrm{e}^{\mathrm{i}z/2} - \mathrm{i} \right.$$

$$\left. + \sum_{m=1}^{\infty} \left[\left(p_1 - A_{11}^{12} \right) \left(A_{1m}^{11} \mathrm{e}^{\mathrm{i}z/2} - \mathrm{i} A_{1m}^{12} \right) + A_{1m}^{12} \mathrm{e}^{\mathrm{i}z/2} - \mathrm{i} A_{1m}^{22} \right] \right] \mathrm{e}^{\mathrm{i}mz}$$

$$\times \left[\left(q_0 - B_{0,-1}^{21} \right) \mathrm{e}^{3\mathrm{i}z/2} - \mathrm{i}\mathrm{e}^{\mathrm{i}z} + \sum_{-\infty}^{m=-1} \left[\left(q_0 - B_{0,-1}^{21} \right) \left(B_{0m}^{22} \mathrm{e}^{3\mathrm{i}z/2} - \mathrm{i} B_{0m}^{21} \mathrm{e}^{\mathrm{i}z} \right) \right. \right.$$

$$\left. \left. + \ B_{0m}^{12} \mathrm{e}^{3\mathrm{i}z/2} - \mathrm{i} B_{0m}^{11} \mathrm{e}^{\mathrm{i}z/2} \right] \mathrm{e}^{-\mathrm{i}mz} \right] \Bigg\}, \tag{3.3}$$

hold [16].

Definition 3.1. The multiplicity of a zero of w in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of Eq. (1.1).

Let, for some $\varepsilon > 0$ and $1/2 \le \delta < 1$,

$$\sum_{n\in\mathbb{Z}} \exp(\varepsilon |n|^{\delta})(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|) < \infty \tag{3.4}$$

hold. For $\delta = 1$ condition (3.4) reduces to

$$\sum_{n\in\mathbb{Z}} \exp(\varepsilon |n|)(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|) < \infty. \tag{3.5}$$

In [16] it is shown that the function w has analytic continuation from the real axis to the lower half-plane under condition (3.5). Using this analytic continuation the following is also proved.

Theorem 3.1. *Under condition (3.5), Eq. (1.1) has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.*

Now let us show that Theorem 3.1 is valid under condition (3.4).

Note that the condition (3.4) is weaker than (3.5). It follows from (2.2), (2.3) and (3.3) that under condition (3.4) the function w is analytic in $\mathbb{C}_+$ and infinitely differentiable on the real axis. But w does not have an analytic continuation from the real axis to the lower half-plane. Consequently under condition (3.4) the finiteness of the discrete spectrum of Eq. (1.1) can not be shown in a way similar to Theorem 3.1.

Eqs. (3.1) and (3.2) show that, in order to investigate the quantitative properties of the discrete spectrum of (1.1), we need to discuss the quantitative properties of the zeros of w in P.

Let

$$M_1 := \{z : z \in P_0,\ w(z) = 0\}, \qquad M_2 := \{z : z \in [0, 4\pi],\ w(z) = 0\}.$$

It follows from (3.1) and (3.2) that

$$\sigma_d = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2},\ z \in M_1 \right\},$$
$$\sigma_{ss} = \left\{ \lambda : \lambda = 2 \sin \frac{z}{2},\ z \in M_2 \right\}.$$

We denote the set of all limit points of M_1 and M_2 by M_3 and M_4, respectively and the set of all zeros of w with infinite multiplicity in P by M_5.

Theorem 3.2. *If* (3.4) *holds, then*

(i) *The set M_1 is bounded and countable.*
(ii) $M_1 \cap M_3 = \emptyset,\ M_1 \cap M_4 = \emptyset,\ M_1 \cap M_5 = \emptyset$
(iii) *The set M_2 is compact and $\mu(M_2) = 0$, where μ denotes the Lebesgue measure in the real axis.*
(iv) $M_3 \subset M_2,\ M_4 \subset M_2,\ M_5 \subset M_2,\ \mu(M_3) = \mu(M_4) = \mu(M_5) = 0.$
(v) $M_3 \subset M_5,\ M_4 \subset M_5.$

Proof. Using (2.2), (2.3) and (3.3) we have

$$w(z) = \left[\prod_{k\in\mathbb{Z}}(-1)^k a_k b_k\right]^{-1} [1 + o(1)], \qquad z \in P_0,\ \operatorname{Im} z \to \infty. \tag{3.6}$$

Eq. (3.6) shows that M_1 is bounded. Since w is analytic in $\mathbb{C}_+$ and is a 4π periodic function we get that M_1 has at most a countable number of elements. This proves (i).

From the uniqueness theorems of analytic functions we obtain (ii)–(iv) [17].

Using the continuity of all derivatives of w on $[0, 4\pi]$ we get (v). □

Lemma 3.3. *Under condition* (3.4) *the inequalities*

$$\sup_{z\in P} |w^{(k)}(z)| \le \mathcal{D}_k, \qquad k = 0, 1, \ldots,$$

hold, where

$$\mathcal{D}_k \le \mathcal{D} d^k k! k^{k\frac{1-\delta}{\delta}}, \tag{3.7}$$

and $\mathcal{D}$ and d are positive constants depending on ε and δ.

Proof. It follows from (2.2), (2.3) and (3.4) that

$$|A_{0m}^{ij}|, |A_{1m}^{ij}| \le C \exp\left(-\frac{\varepsilon}{2}\left|\frac{m}{2}\right|^{\delta}\right), \qquad i, j = 1, 2;\ m = 1, 2, \ldots, \tag{3.8}$$

and

$$|B_{0m}^{ij}|, |B_{1m}^{ij}| \le C \exp\left(-\frac{\varepsilon}{2}\left|\frac{m}{2}\right|^{\delta}\right), \qquad i, j = 1, 2;\ m = -1, -2, \ldots, \tag{3.9}$$

hold. Also we have from (3.3), (3.8) and (3.9) that

$$\sup_{z\in P} |w^{(k)}(z)| \le \mathcal{D}_k, \qquad k = 0, 1, \ldots,$$

where

$$\mathcal{D}_k = 2^k C \sum_{m=1}^{\infty} m^k \exp\left(-\frac{\varepsilon}{2} m^{\delta}\right), \qquad k = 0, 1, \ldots.$$

Now we get the estimate

$$\mathcal{D}_k \le 2^k C \int_0^{\infty} t^k \exp\left(-\frac{\varepsilon}{2} t^{\delta}\right) \mathrm{d}t = 2^k C \left(\frac{2}{\varepsilon}\right)^{k/\delta} \left(\frac{2}{\varepsilon}\right)^{1/\delta} \frac{1}{\delta} \int_0^{\infty} t^{\frac{k+1}{\delta}-1} \mathrm{e}^{-t} \mathrm{d}t. \tag{3.10}$$

Let us denote the integer part of $(\frac{k+1}{\delta} - 1)$ by ν. If we apply the partial integration ν times in (3.10) we find

$$\mathcal{D}_k \leq 2^k C \left(\frac{2}{\varepsilon}\right)^{\frac{k+1}{\delta}} \left(\frac{2}{\varepsilon}\right)^{1/\delta} \frac{1}{\delta} \left(\frac{k+1}{\delta}\right)^{\frac{k+1}{\delta}-1} \int_0^\infty t^{\frac{k+1}{\delta}-(\nu+1)} \mathrm{e}^{-t} \mathrm{d}t. \tag{3.11}$$

Using the inequalities

$$\left(1+\frac{1}{k}\right)^{k/\delta} < \mathrm{e}^{1/\delta},\ k^k < k!\mathrm{e}^k,\ (1+k)^{\frac{1}{\delta}-1} < \mathrm{e}^{k/\delta}$$

we get

$$\mathcal{D}_k \leq \mathcal{D} d^k k! k^{k\frac{1-\delta}{\delta}}, \qquad k = 0, 1, \ldots,$$

by (3.11), where $\mathcal{D}$ and d are positive constants depending on ε and δ. □

We will use the following uniqueness theorem for the analytic functions, to prove the next result.

Theorem 3.4 ([3]). *Let us assume that a 4π periodic function ϕ is analytic in $\mathbb{C}_+$, all of its derivatives are continuous in $\overline{\mathbb{C}}_+$ and*

$$\sup_{z\in P} |\phi^{(k)}(z)| \leq Q_k, \qquad k = 0, 1, \ldots,$$

and the set $G \subset [0, 4\pi]$ with $\mu(G) = 0$ is the set of all zeros of the function ϕ with infinite multiplicity in P. If

$$\int_0^\alpha \ln T(s) \mathrm{d}\mu(G_s) = -\infty, \tag{3.12}$$

holds, where

$$T(s) = \inf_k \frac{Q_k s^k}{k!}, \qquad k = 0, 1, \ldots,$$

and $\mu(G_s)$ is the Lebesgue measure of the s-neighborhood of G, and $\alpha \in [0, 4\pi]$ is an arbitrary constant, then $\phi \equiv 0$ in $\overline{\mathbb{C}}_+$.

Lemma 3.5. *If (3.4) holds, then $M_5 = \emptyset$.*

Proof. The function w satisfies all conditions of Theorem 3.4 except (3.12). But w is not identically equal to zero. In this case the function w satisfies the condition

$$\int_0^\alpha \ln T(s) \mathrm{d}\mu(M_{5,s}) > -\infty, \tag{3.13}$$

instead of (3.12), where

$$T(s) = \inf_k \frac{\mathcal{D}_k s^k}{k!}, \qquad k = 0, 1, \ldots,$$

and $\mu(M_{5,s})$ is the Lebesgue measure of the s-neighborhood of M_5, and $\mathcal{D}_k$ is defined by (3.7). Substituting (3.7) in the definition of $T(s)$, we get

$$T(s) = \mathcal{D} \exp\left[-\frac{1-\delta}{\delta} \mathrm{e}^{-\frac{1}{1-\delta}} d^{-\frac{1}{1-\delta}} s^{-\frac{1}{1-\delta}}\right]. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\int_0^\alpha s^{-\frac{1}{1-\delta}} \mathrm{d}\mu(M_{5,s}) < \infty. \tag{3.15}$$

Since $\frac{\delta}{1-\delta} \geq 1$, consequently (3.15) holds for arbitrary s if and only if $\mu(M_{5,s}) = 0$ or $M_5 = \emptyset$. □

Theorem 3.6. *Under condition* (3.4) *Eq.* (1.1) *has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. To be able to prove the theorem we have to show that the function w has a finite number of zeros with finite multiplicities in P.

From Theorem 3.2 and Lemma 3.5 we get that $M_3 = M_4 = \emptyset$. So the bounded sets M_1 and M_2 have no limit points, i.e., the function w has only a finite number of zeros in P. Since $M_5 = \emptyset$, these zeros are of finite multiplicity. □

From Theorem 3.6, we get that the weakest condition which guarantees the finiteness of eigenvalues and spectral singularities of Eq. (1.1) is

$$\sum_{n\in\mathbb{Z}} \exp\left(\varepsilon\sqrt{|n|}\right)(|1-a_n| + |1+b_n| + |p_n| + |q_n|) < \infty, \tag{3.16}$$

for some $\varepsilon > 0$.

The condition (3.16) indicates that the open problem given in [16] has been solved positively.

From Theorem 3.6 and (3.16) we get the following theorem for the discrete Dirac system.

Theorem 3.7. *If*

$$\sum_{n\in\mathbb{Z}} \exp\left(\varepsilon\sqrt{|n|}\right)(|p_n| + |q_n|) < \infty, \qquad \varepsilon > 0$$

holds, then the discrete Dirac system (1.2) *has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Note that the same result has been obtained in [4] under the stronger assumption

$$\sup_{n\in\mathbb{Z}} \left[\exp(\varepsilon|n|)(|p_n| + |q_n|)\right] < \infty,$$

for some $\varepsilon > 0$.

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